Supplementary Material for Paper #684, NIPS 2007

Lemma 1. *When the vertical and horinzontal shifts a and b are chosen uniformly at random from $0, 1, \ldots, l$, then*

$$E_{a,b}[d_{a,b,m}(s,s')] \leq (1 + \frac{2\log L}{m})d(s,s').$$

Proof. We will upper-bound the expectation by the desired value. The straight line path from s to s' crosses the snit grid at most $2d(s,s')$ times. A portal-respecting path can be obtained by moving each of these crossings to the nearest portal (see figure 2), which will increase the size of the path by at most the inter-portal distance for that line. For a line at level i, this distance is $\frac{L}{m2^{i+1}}$. By Proposition 1, the probability that the line is at level i is $\frac{2^{i+1}}{L}$. Hence the expected length of the detour is at most:

$$\sum_{i=0}^{\log L-1} \frac{2^{i+1}}{L} \cdot \frac{L}{m2^{i+1}} = \frac{\log L}{m}$$

This is true for each of the $2d(s,s')$ crossings, and therefore by linearity of expectations, the expected increasing in moving all the crossings to portals is $2d(s,s')\frac{\log L}{m}$. □

Theorem 1. *If L is the sidelength of the enclosing box and m is the portal parameter, then for any state s_0,*

$$E_{a,b}[V^*_{a,b,m}(s_0)] \leq (1 + 2\frac{\log L}{m})V^*(s_0)$$

Proof. The optimal value of the arbitrary state s_0 is given by:

$$V^*(s_0) = \sum_{i=0}^{\infty} \gamma^i d(s_i, s_{i+1})$$

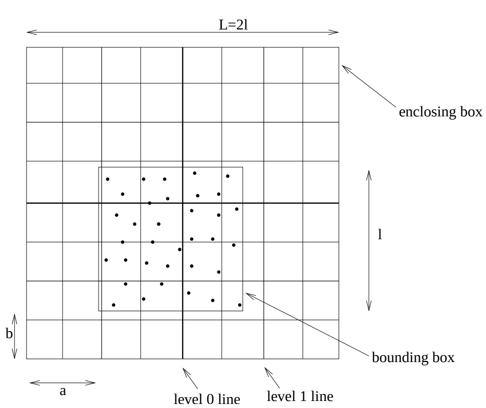

Figure 1: A randomized dissection. The nodes are all contained within the bounding box. Random vertical and horizontal shifts a and b are chosen for the enclosing box, which is then recursively partitioned.

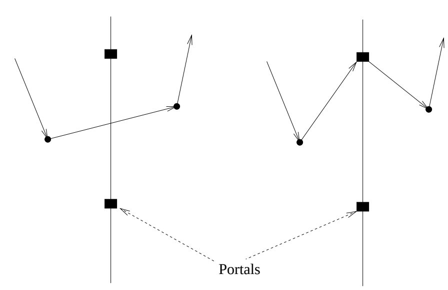

Figure 2: A portal respecting path must take a detour through a portal every time it crosses a gridline. The extra cost of doing so is at most the inter-portal distance.

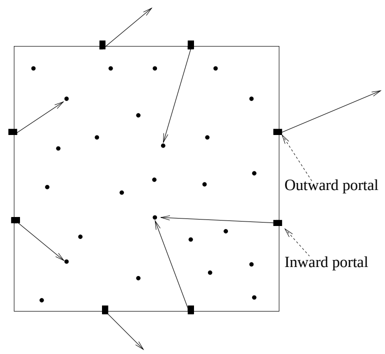

Figure 3: The portals on the boundary of a dissection square point either inward or outward. Those that point outward have their values fixed and taken from $\mathbb{V}$. This defines an *interface* to the square.

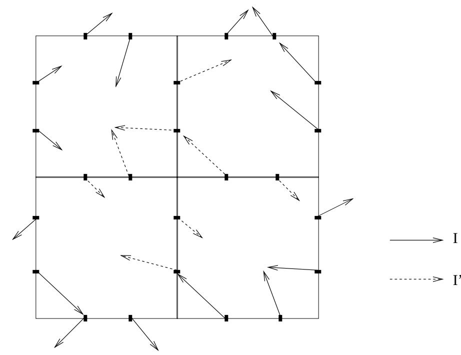

Figure 4: The patching procedure. The interface I for the portals on the outer boundary is given. When an interface I' is chosen for the portals along the bisecting lines, the interfaces for the child squares are completed and its policy can be looked up in the interface table.

where $s_{i+1} = \pi^*(s_i)$.

Let $\pi_{a,b,m}$ be the portal-respecting policy obtained from π^* as follows: Make each edge of π^* portal-respecting, and then set $\pi_{a,b,m}(s)$ to be the next node along the portal-respecting path from s to $\pi^*(s)$. If there are one or more of these paths through portal p, set $\pi_{a,b,m}(p)$ to be that next node having the least value according to V^*. Notice that the last step can only decrease the value of the policy whereas the first step increases the value by a factor of at most $2\frac{\log L}{m}$. For portals without any such path through them, make an arbitrary choice for $\pi_{a,b,m}(p)$.

Since the portals are undiscounted, an edge from u to v where u and v are nodes in the original instance can be treated as a single edge with cost $d_{a,b,m}(u, v)$. We have,

$$\begin{aligned} V^*_{a,b,m}(s_0) &\leq V^{\pi_{a,b,m}}(s_0) \\ &\leq \sum_{i=0}^{\infty} \gamma^i d_{a,b,m}(s_i, s_{i+1}) \\ &= \sum_{i=0}^{\infty} \gamma^i d(s_i, s_{i+1})[1 + 2\frac{\log L}{m}] \\ &= V^*(s_0)[1 + 2\frac{\log L}{m}] \quad \square \end{aligned}$$

Theorem 2. *The Dynamic Program outlined in section 3 when run on a problem instance with random shifts a and b and portal parameter m, returns a portal-respecting policy π s.t. $V^\pi \leq (1+\delta)^h V^*_{a,b,m}$, where $V^*_{a,b,m}$ is the optimal portal-respecting value for a, b and m, and h is the depth of the dissection tree ($h = O(n^2)$).*

Proof. We prove the following statement by induction for all $i, 0 \leq i \leq h-1$: For all squares S and interfaces I at level i, the value V_I returned by our algorithm is bounded by

$$V_I \leq (1+\delta)^{h-i} V^*_I$$

where V^*_I is the optimal value for interface I. The theorem then follows by using a similar proof for the root square which has no portals on its boundary.

Let S be a square at level i, subdivided as in figure 4 into 4 squares of level $i+1$, and let I be an interface for S. For every portal p on the level i lines bisecting S, define $V^t_I(p) = (1+\delta)^k V_{min}$ such that

$$(1+\delta)^{k-1} V_{min} \leq V^*_I(p) \leq (1+\delta)^k V_{min}$$

Note that this ensures that $V_I^t \leq (1+\delta)V_I^*$. Let I' be the interface for the portals along the level i lines that have these values (and a choice for the direction at each portal corresponding to π_I^*. Then $I \cup I'$ defines an interface for the sub-squares at level $i+1$ inside S.

Let s be a node inside one of these level $i+1$ squares S'. There are two cases:

Case 1: The path followed by the optimal policy π_I^* starting from s stays completely within S'. In this case, the value assumed by the portals are irrelevant. By the induction hypothesis we have,

$$V_I(s) = V_{I \cup I'}(s) \leq (1+\delta)^{h-i-1} V_I^*(s)$$

Case 2: Let p be the first portal on the boundary of S' reached along the path P_s followed by the optimal policy π_I^* starting from s. Then,

$$V_{I \cup I'}^*(s) = d_{P_s}(s,p) + V_{I \cup I'}^*(p)$$

where $d_{P_s}(s,p)$ is the distance along the path P_s from s to p.

The algorithm returns a policy for interface I that is the minimization over all interfaces I'. Thus,

$$\begin{aligned} V_I(s) &\leq V_{I \cup I'}(s) \\ &\leq (1+\delta)^{h-i-1} V_{I \cup I'}^*(s) \quad \text{(Induction hypothesis)} \\ &\leq (1+\delta)^{h-i-1}[d_{P_s}(s,p) + V_{I \cup I'}^t(p)] \end{aligned}$$

The last step follows because $V_{I \cup I'}^*$ is the minimum value obtainable for the sub-problem $(S', I \cup I')$ for which we can obtain $d_{P_s}(s,p) + V_{I \cup I'}^t(s)$ by moving along P_s. Continuing,

$$\begin{aligned} V_I(s) &\leq (1+\delta)^{h-i-1}[d_{P_s}(s,p) + (1+\delta)V_I^*(p)] \\ &\leq (1+\delta)^{h-i}[d_{P_s}(s,p) + V_I^*(p)] \\ &= (1+\delta)^{h-i} V_I^*(s) \end{aligned}$$

□