1. Propositional Logic

Convert the following propositional formulae into Conjunctive Normal Form and for each one, characterize the set of satisfying assignments:

(a) \((x_1 \land x_2 \land x_3) \lor (\overline{x_2} \land x_3) \lor (x_1 \land \overline{x_4})\)

\(\text{Ans. Expanding:}\)
\[
\begin{align*}
&(x_1 \lor x_2 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_3 \lor x_1) \land (x_1 \lor x_2 \lor \overline{x_4}) \\
&\land (x_2 \lor x_3 \lor x_1) \land (x_2 \lor \overline{x_3} \lor \overline{x_4}) \\
&\land (x_3 \lor \overline{x_2} \lor x_1) \land (x_3 \lor \overline{x_2} \lor \overline{x_4}) \\
&\land (x_4 \lor \overline{x_2} \lor x_1) \land (x_4 \lor \overline{x_2} \lor \overline{x_4})
\end{align*}
\]

Simplifying:
\[
\begin{align*}
&(x_1 \lor x_2 \lor \overline{x_4}) \land (x_1 \lor x_3 \lor \overline{x_4}) \land (x_2 \lor x_3 \lor x_1) \land (x_2 \lor x_3 \lor \overline{x_4}) \\
&\land (x_3 \lor x_1) \land (x_3 \lor \overline{x_4}) \\
&\land (x_4 \lor \overline{x_4})
\end{align*}
\]

Removing redundant clauses:
\[
(x_1 \lor x_2 \lor \overline{x_4}) \land (x_3 \lor x_1) \land (x_4 \lor \overline{x_4})
\]

Solutions: \(x_3 = 1\) and at least one of \(x_1, x_2, x_4\) false or \(x_1 = 1, x_4 = 0, x_2, x_3 = \text{any.}\)

The original problem (with the typo) was \((\overline{x_1} \land x_2 \land x_3) \lor (\overline{x_2} \lor x_3) \lor (x_1 \land \overline{x_4})\). The CNF for this is \((x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_2} \lor x_3 \lor x_4)\). A satisfying assignment for this will have either \(x_2\) false or \(x_3\) true or both \(x_1\) and \(x_4\) true.

(b) \((\overline{x_1} \land x_2) \lor x_3 \land x_4 \lor \ldots \lor x_{2n}\)

\(\text{Ans. By Demorgan’s law,}\)
\[
\begin{align*}
\overline{x_1} \land x_2 &= x_1 \lor x_2 \\
\overline{x_1} \lor x_2 \land \overline{x_3} &= \overline{x_1} \land x_2 \land \overline{x_3} \\
\overline{x_1} \land x_2 \land \overline{x_3} \land x_4 &= \overline{x_1} \lor x_2 \lor x_3 \lor x_4
\end{align*}
\]
Continuing in this way we get,

\[ x_1 \land x_2 \lor x_3 \land x_4 \lor \ldots \land x_{2n} = x_1 \lor x_2 \land x_3 \lor x_4 \lor \ldots \lor x_{2n} \]

The set of solutions is any assignment where any \( x_{2i} \) is false or any \( x_{2i+1} \) is true for \( 0 \leq i \leq n \).

(c) \( x_1 \oplus x_2 \oplus \ldots \oplus x_n \)

**Ans.** Let \( F = x_1 \oplus x_2 \oplus \ldots \oplus x_n \).

The set of satisfying assignments of \( F \) is the set of all assignments with an odd number of variables true. Therefore, the set of satisfying assignments of \( F \) is the set of all assignments with an even number of variables true.

We can write \( F \) in Disjunctive normal form as follows:

\[ F = \bigvee_{j \in \text{Even}(T(n))} T_j \]

where \( \text{Even}(T(n)) \) is the set of all terms (conjunction of \( n \) literals) with an even number of variables un-negated. e.g. for \( n = 3 \), \( F = (x_1 \land x_2 \land \lnot x_3) \lor (x_1 \land \lnot x_2 \land x_3) \lor (\lnot x_1 \land x_2 \land x_3) \lor (\lnot x_1 \land \lnot x_2 \land \lnot x_3) \).

By the principle of duality,

\[ F = \bigwedge_{j \in \text{Odd}(C(n))} C_j \]

where \( \text{Odd}(C(n)) \) is the set of all clauses (disjunction of \( n \) literals) with an even number of variables negated. e.g. for \( n = 3 \), \( F = (\lnot x_1 \lor \lnot x_2 \lor x_3) \land (\lnot x_1 \lor x_2 \lor \lnot x_3) \land (x_1 \lor x_2 \lor \lnot x_3) \).

2. Linear Algebra

(a) Let \( B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \) and \( C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix} \). Find a non-zero \( 2 \times 2 \) matrix \( A \) s.t. \( AB = AC \) and \( A^T B = A^T C \).

**Ans.** Let \( A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \). Since \( AB = AC \),

\[
\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}
\]

Thus we have,

\[
\begin{aligned}
2x + 3y &= -2x + 5y \\
x + 2y &= 7x - y \\
2z + 3w &= -2z + 5w \\
z + 2w &= 7z - w
\end{aligned}
\]
Solving, we get \( y = 2x, w = 2x \). From \( A^T B = A^T C \), we get \( z = 2x, w = 2y \). (You do not have to repeat the calculation to see this.)

One matrix that satisfies these constraints is \[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\]

(b) What is the rank of the following matrix?

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & -3 & 4 \\
2 & -1 & 7
\end{bmatrix}
\]

What is its inverse? What implications does this have for the following system of equations:

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= 4 \\
x_1 - 3x_2 + 4x_3 &= 5 \\
2x_1 - x_2 + 7x_3 &= 6
\end{align*}
\]

\textbf{Ans.} \((1, 2, 3) + (1, -3, 4) = (2, -1, 7)\), but \((1, 2, 3) \neq k(1, -3, 4)\) for any \(k \in \mathbb{R}\). Since at most two row vectors are linearly independent, the rank of the matrix is 2. Therefore it does not have an inverse. The rank of the augmented matrix:

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & -3 & 4 & 5 \\
2 & -1 & 7 & 6
\end{bmatrix}
\]

is 3. Since this is not equal to the rank of the original matrix, the system of equations has no solution.

(c) A square matrix \( A \) is \textit{diagonalizable} if there is a non-singular matrix \( T \) and a diagonal matrix \( D \) s.t. \( T^{-1}AT = D \). Show that an \( n \times n \) matrix \( A \) is diagonalizable if and only if it has \( n \) linearly independent eigenvectors.

\textbf{Ans.} Suppose \( A \) is diagonalizable with \( T \) and \( D \) being the corresponding non-singular and diagonal matrix respectively. Pre-multiplying by \( T \) on both sides gives \( AT = TD \). Let \( d_1, \ldots, d_n \) be the entries along the diagonal of \( D \) and \( T_1, \ldots, T_n \) be the column vectors of \( T \). Therefore, we have for each \( 1 \leq i \leq n, AV_i = d_iV_i \) (verify this by multiplying the matrices and comparing the column vectors.) Therefore \( V_1, \ldots, V_n \) are eigenvectors of \( A \) and \( d_1, \ldots, d_n \) are the corresponding eigenvalues. Since \( T \) is non-singular, \( V_1, \ldots, V_n \) are all different and linearly independent.

To show the opposite direction, take \( T \) and \( D \) to be the matrices with eigenvectors of \( A \) as columnvectors and eigenvalues along the diagonal, respectively. Then it is easy to show that \( AT = TD \). Since the eigenvectors are linearly independent, \( T \) is non-singular. Therefore it has an inverse and pre-multiplying by \( T^{-1} \) gives \( T^{-1}AT = D \).
3. Probability theory

(a) Suppose that the scores of Homework 0 can be described by a Gaussian Distribution with mean $\mu = 85$ and variance $\sigma^2 = 15$. What is the probability that a student has a grade greater than 95?

**Ans.** Converting to a normal distribution (Gaussian with $\mu = 0$, $\sigma = 1$),

$$z = \frac{95 - 85}{\sqrt{15}} = 2.581989$$

Consulting the probability tables for the normal distribution we get

$$P(Z > 2.581989) = 0.004912$$

(b) The Department of Computer Science at UIUC believes that students can be divided into two groups - those who are interested in A.I. and those who are not interested in A.I. Statistics show that students who like A.I. will enroll in CS440 with probability 0.8 whereas the other students will do so with probability 0.3. If we assume that 60 percent of all students are interested in A.I., what is the probability that a student who enrolled in CS440 is interested in A.I.?

**Ans.** $P(AI) = 0.6 \Rightarrow P(\neg AI) = 0.4$. Also, $P(CS440|AI) = 0.8$, $P(CS440|\neg AI) = 0.3$.

$$P(AI|CS440) = \frac{P(AI \cap CS440)}{P(CS440)} = \frac{P(CS440|AI)P(AI)}{P(CS440|AI)P(AI) + P(CS440|\neg AI)P(\neg AI)} = \frac{0.8 \cdot 0.6}{0.8 \cdot 0.6 + 0.3 \cdot 0.4} = \frac{0.8}{0.8} = 0.8$$

(c) Suppose we wish to generate samples from a continuous strictly increasing cumulative distribution function $F$ on the interval $[a, b]$. Show how you can do so by using a uniform generator on the interval $[c, d]$. (Note: There are many ways to do this. Try to solve this problem by using the fact that $F$ is strictly increasing.)

**Ans.** First we need to transform a uniformly generated random variable $X$ in the range $[c, d]$ to the range $[a, b]$. This can be done by the following transformation:

$$T(X) = (X - c) \cdot \frac{b - a}{d - c} + a$$

If $X$ is uniform in the range $[c, d]$, $T(X)$ will now be uniform in the range $[a, b]$. 
Now we need to show how to use this to sample according to the probability distribution with cdf $F$ in the range $[a, b]$. First, observe that $F$ is strictly increasing $\Rightarrow F$ has an inverse $F^{-1}$.

First, we take a sample $X$ from a uniform distribution on $[0, 1]$ using the technique above. Since $X$ is a uniform random variable,

$$P(0 \leq X \leq x) = \frac{x - 0}{1} = x$$

Then we return the following value as our answer:

$$Z = F^{-1}(X)$$

Since the domain of $F$ is $[a, b]$, $Z$ will be in this range. Also, the cumulative distributive function for $Z$ is

$$P(a \leq Z \leq z) = P(a \leq F^{-1}(X) \leq z) = P(0 \leq X \leq F(z)) \quad \text{(Because $F$ is an increasing function)}$$

$$= F(z)$$

Therefore $Z$ has exactly the distribution we wanted.